The endoscopic fundamental lemma for unitary Friedberg-Jacquet periods

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Relative Endoscopy?

Relative Langlands program

- Periods of automorphic forms
- Spherical varieties
- Connections to L-functions

$$\int_{\mathbb{R}_{>0}} f(iy)dy = L(1, f)$$

Endoscopy

- Stabilization of trace formula
- Structure of L-packets

$$\mathrm{SO}(\gamma', \mathbf{1}_{H_\kappa(\mathbb{Z}_p)}) = \Delta(\gamma, \gamma')\mathrm{O}^\kappa(\gamma, \mathbf{1}_{H(\mathbb{Z}_p)})$$

Outline

1. Consider a case in which a theory of relative endoscopy is useful.
2. Explain the global goals and comparison of relative trace formulae.
3. Consider local orbital integrals and state (relative) fundamental lemma
4. (If there's time) Outline the proof, and explain endoscopic symmetric spaces.

Periods of automorphic forms

- Fix an imaginary quadratic extension $E/\mathbb{Q}$, adele rings $\mathbb{A}_E/\mathbb{A}$
- Let W_1 and W_2 be two n dimensional E-Hermitian spaces.
- $U(W_1)$, $U(W_2)$ the associated unitary groups.

$$\begin{array}{ccc} U(W_1) \times U(W_2) & \subset & U(W_1 \oplus W_2) \\ H & \subset & G \end{array}$$

$U(W_1) \times U(W_2)$-distinguished forms

$$\begin{array}{ccc} U(W_1) \times U(W_2) & \subset & U(W_1 \oplus W_2) \\ H & \subset & G \end{array}$$

Distinguished forms

Study automorphic forms $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that

$$\mathcal{P}_H(\varphi) := \int_{[H]} \varphi(h)dh \neq 0,$$

where $[H] := H(\mathbb{Q})\backslash H(\mathbb{A})$. We refer to these as **unitary Friedberg-Jacquet periods.**

Geometric interpretation

With appropriate choices (ie: compatible Shimura data), the embedding

$$(U(W_1) \times U(W_2)) \subset U(W_1 \oplus W_2)$$
$$\Downarrow$$
$$\mathcal{S}h_{GU(W_1)} \times \mathcal{S}h_{GU(W_2))} \subset \mathcal{S}h_{GU(W_1 \oplus W_2)}$$

gives rise to special cycles in the Shimura variety $\mathcal{S}h_{GU(W_1 \oplus W_2)}$.

Low-rank Examples

Example

$\underline{n = 1:}$ $U(1) \times U(1) \subset U(1,1) \Rightarrow$ Heegner point in a Modular curve

Example

$\underline{n = 3:}$

$$U(1,2) \times U(2,1) \subset U(3,3)$$

product of Picard modular surfaces in 9-dimensional Shimura variety

Low-rank Examples

Example

$\underline{n=1}$: $U(1)\times U(1)\subset U(1,1)\Rightarrow$ Heegner point in a Modular curve

Example

$\underline{n=3}$:,

$$U(1,2)\times U(1,2)\subset U(2,4)$$

product of Picard modular surfaces in 8-dimensional Shimura variety

- Can choose information so that cycles are at middle codimension (geometric or arithmetic)

Applications

- Study cohomology classes dual to these cycles in $H^*_{\text{ét}}(\mathcal{S}h_{GU(W_1 \oplus W_2)})$.
- Relate to information on poles of Zeta function of $\mathcal{S}h_{GU(W_1 \oplus W_2)}$.

Translating back to automorphic representations

Determine automorphic representations π of $G = U(W_1 \oplus W_2)$ that are $U(W_1) \times U(W_2)$-distinguished.

(Graham-Shah, '20) Construct Euler systems in the setting

$$U(1, 2k) \times U(0, 2k+1) \subset U(1, 4k+1)$$

partial results towards Bloch-Kato conjectures.

Friedberg-Jacquet: Linear periods

- Consider the linear groups associated to W_1, W_2
- $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \subset \mathrm{GL}(W_1 \oplus W_2)$

Characterized cuspidal aut reps Π such that

$$\int_{[\mathrm{GL}(W_1)\times \mathrm{GL}(W_2)]} \varphi(h)dh \neq 0$$

for a cusp form $\varphi \in \Pi$.

Theorem (Friedberg-Jacquet, '95)

Π is $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-distinguished if and only if

1. $L(\frac{1}{2}, \Pi) \neq 0$
2. $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$

Base change to $\mathrm{GL}(W)$

$$\begin{array}{ccc} \mathrm{GL}(W_1) \times \mathrm{GL}(W_2) \subset \mathrm{GL}(W_1 \oplus W_2) & & \Pi \\ & \text{Base change} & \uparrow \\ U(W_1) \times U(W_2) \subset U(W_1 \oplus W_2) & & \pi \end{array}$$

Conjecture (Getz-Wambach, rough form)

Suppose π and $\Pi = BC(\pi)$ are both cuspidal. Then

$$\pi \text{ is } U(W_1) \times U(W_2)\text{- dist.} \iff \Pi \text{ is } \mathrm{GL}(W_1) \times \mathrm{GL}(W_2)\text{- dist.}$$

(Pollack-Wan-Zydor ‘19) ($\Rightarrow$) holds under certain circumstances.

Method: Comparison of relative trace formulas

- Let's ignore **ALL** analytic complications:

For any $f \in C_c^\infty(G(\mathbb{A}))$, consider the distribution

$$J(f) := \int_{[H]} \int_{[H]} \left(\sum_{\gamma \in G(\mathbb{Q})} f(h^{-1}\gamma g) \right) dg dh.$$

RTF

$$\sum_{\pi} c(\pi) \sum_{\varphi} |\mathcal{P}_H(\varphi)|^2 \lambda_\pi(f) \approx J(f) \approx \sum_{\gamma} a(\gamma) \mathrm{O}(\gamma, f),$$

where π sums over $U(W_1) \times U(W_2)$-distinguished reps and γ sums over orbits.

Geometric side

Here γ ranges over $H(F) \times H(F)$-orbits on $G(F)$, and

$$\mathrm{O}(\gamma, f) := \int_{H_\gamma(\mathbb{A})\backslash H(\mathbb{A})\times H(\mathbb{A})} f(g^{-1}\gamma h)dgdh$$

is an orbital integral over the $H(\mathbb{A}) \times H(\mathbb{A})$-orbit of γ,

$$H_\gamma = \{(g, h) \in H \times H : g^{-1}\gamma h = \gamma\}.$$

is the stabilizer of γ

The linear relative trace formula

For $f' \in C_c^\infty(\mathrm{GL}(W_1 \oplus W_2)_{\mathbb{A}_E})$,

$$I(f') := \int_{[\mathrm{GL}(W_1)\times \mathrm{GL}(W_2)]} \int_{[U(\tilde{W})]} \left(\sum_{x \in \mathrm{GL}(W_1 \oplus W_2)_E} f'(g^{-1}xh) \right) dhdg$$

RTF

$$\sum_{\Pi} [\text{spectral (period) terms}] \approx I(f') \approx \sum_{\delta} c(\delta)\mathrm{TO}(\delta, f')$$

Only forms that are $\mathrm{GL}(W_1) \times \mathrm{GL}(W_2)$-dist. and base changes from $U(W_1 \oplus W_2)$ appear.

Desired comparison

Comparison of RTFs

Find a matching of functions $f' \leftrightarrow f$ such that

$$I(f') = J(f),$$

by comparing geometric sides of RTFs.

That is:

- match orbits $\gamma \leftrightarrow \delta$,
- match orbital integrals

$$\mathrm{O}(\gamma, f) = \mathrm{TO}(\delta, f').$$

Problem: There is **no natural matching of rational orbits**, only stable orbits.

Stable versus Rational orbits

(Getz-Wambach) There is a natural *norm map*

$$Nm : \{\text{Orbits in } \mathrm{GL}(W_1 \oplus W_2)\}(\overline{\mathbb{Q}}) \to H\backslash G/H(\overline{\mathbb{Q}})$$

over $\overline{\mathbb{Q}}$ (stable orbits), but not over $\mathbb{Q}$ (rational orbits).

- Stable orbits decompose into several rational orbits.
- Need stable versions of orbital integrals to compare.

Need a decomposition

$$J(f) = \underbrace{SJ(f)}_{\text{Stable RTF}} + (\text{error terms})$$

This is essentially a local problem.

Local problem

Simplifying assumptions:

- Let E/F denote an unramified quadratic extension of p-adic fields.
- $V_n = W_1 = W_2$ be a **split** Hermitian space over E of dimension n.

ie: there is a lattice $\Lambda_n \subset V_n$ that is self-dual with respect to the Hermitan form.

- This gives hyperspecial subgroups

$$U(\Lambda_n) \subset U(V_n), \text{ and } U(\Lambda_n \oplus \Lambda_n) \subset U(V_n \oplus V_n)$$

The symmetric space

Set $H(F) = U(V_n) \times U(V_n)$. Studying

$$H(F) \times H(F)\text{-orbital integrals on } G(F)$$

reduces to studying $H(F)$-orbital integrals of on the symmetric space

$$\mathcal{Q}_n(F) = (G/H)(F) \subset G(F).$$

Our choice of self-dual lattice $\Lambda_n \subset V_n$ gives the compact open subset

$$\mathcal{Q}_n(\mathcal{O}_F) = G(\mathcal{O}_F)/H(\mathcal{O}_F),$$

where $\mathcal{O}_F \subset F$ is the ring of integers.

Local geometric stabilization: orbital integrals

The group $H(F) = U(V_n) \times U(V_n)$ acts on $\mathcal{Q}(F)$ via conjugation.

Orbital integral

For $f \in C_c^\infty(\mathcal{Q}(F))$, and $X \in \mathcal{Q}(F)$ (regular) semi-simple, the *relative orbital integral* if given by

$$\mathrm{O}(X, f) = \int_{H_X(F)\backslash H(F)} f(g^{-1}Xg)dg,$$

where $H_X(F)$ is the stabilizer (a rank n torus).

For the **fundamental lemma**, we care about $f = \mathbf{1}_{\mathcal{Q}(\mathcal{O}_F)}$.

Stable orbital integral

Two types of orbit

- **rational orbits:** $X' = gXg^{-1}$ for $g \in H(F)$
- **stable orbit**: $X' = gXg^{-1}$ for $g \in H(\overline{F})$.

Stable orbital integral

We set

$$\mathrm{SO}(X, f) := \sum_{X \sim_{st} X'} \mathrm{O}(X', f).$$

where the sum is over rational orbits in the stable orbit of X.

- These may be compared to (twisted) orbital integrals on the linear side.

What's the difference?: κ-orbital integrals

These rational orbits are parameterized by cohomology classes

$$inv(X, X') \in H^1(F, H_X)(\cong (\mathbb{Z}/2\mathbb{Z})^r).$$

Definition

For any character $\kappa : H^1(F, T_X) \to \mathbb{C}^\times$, define the **$\kappa$-orbital integral** to be

$$\mathrm{O}^\kappa(X, f) := \sum_{X \sim_{st} X'} \kappa(inv(X, X'))\mathrm{O}(X', f).$$

When $\kappa = 1$, we have $\mathrm{O}^\kappa = \mathrm{SO}$ is the stable orbital integral.

Fourier inversion on $H^1(F, H_X)$

$$\mathrm{O}(X,f) = c\left(\mathrm{SO}(X,f) + \sum_{\kappa\neq 1} \mathrm{O}^{\kappa}(X,f)\right)$$

Local manifestation of stability issue: for $F \in C_c^\infty(G(\mathbb{A}))$, the RTF decomposes

$$J(F) = \underbrace{SJ(F)}_{\text{Stable part of RTF}} + \underbrace{\sum_{\kappa} J^{\kappa}(F)}_{\text{error terms}}$$

Only $SJ(f)$ can be compared to the linear side.

Problem of geometric stabilization

Find groups of smaller dimension H_κ acting on varieties X_κ so that

$$\kappa\text{-orbital integrals of } (H(F), \mathcal{Q}(F))$$

may be expressed in terms of

$$\text{stable orbital integrals of } (H_\kappa(F), X_\kappa(F)).$$

Unramified endoscopic spaces

- Decompose $V_n = V_a \oplus V_b$ into (split) Hermitian subspaces, where $n = a + b$.
- Consider
$$\mathcal{Q}_a \times \mathcal{Q}_b$$
where
$$\mathcal{Q}_a = U(V_a \oplus V_a)/U(V_a) \times U(V_a), \ldots$$
- $U(V_a) \times U(V_a)$ and $U(V_b) \times U(V_b)$ acts on
$$\mathcal{Q}_a(F) \times \mathcal{Q}_b(F).$$

This gives an **unramified elliptic endoscopic space** for $(U(V_n) \times U(V_n), \mathcal{Q}_n(F))$.

Relative endoscopy

Lemma

There exists a matching of stable closed orbits

$$\{X \in \mathcal{Q}(F)\} \leftrightarrow \{(X_a, X_b) \in \mathcal{Q}_a(F) \times \mathcal{Q}_b(F)\}.$$

- There exists a good notion of transfer factor

$$\Delta : \mathcal{Q}^{rrs}(F) \times (\mathcal{Q}_a \times \mathcal{Q}_b)^{rrs}(F) \longrightarrow \mathbb{C}$$

- For many test functions $f \in C_c^\infty(\mathcal{Q}(F))$, there exists $f^{a,b}$ such that

$$\mathrm{SO}((X_a, X_b), f^{a,b}) = \Delta(X, (X_a, X_b))\mathrm{O}^\kappa(X, f).$$

The fundamental lemma for the unit function

Theorem (L)

For any regular semi-simple $X \in \mathcal{Q}(F)$ and $\kappa \in H^1(F, H_X)^*$, there is a decomposition $V_n = V_a \oplus V_b$ and unique stable orbit of elements

$$(X_a, X_b) \in \mathcal{Q}_a(F) \times \mathcal{Q}_b(F)$$

such that

$$\mathrm{SO}((X_a, X_b), \mathbf{1}_{\mathcal{Q}_a(\mathcal{O}_F) \times \mathcal{Q}_b(\mathcal{O}_F)}) = \Delta(X, (X_a, X_b)) \mathrm{O}^{\kappa}(X, \mathbf{1}_{\mathcal{Q}_n(\mathcal{O}_F)}).$$

Crucial for global applications: $f \in C_c^{\infty}(\mathcal{Q}(\mathbb{A}))$

$$f = \prod_{p \in S} f_p \times \prod_{p \notin S} \mathbf{1}_{\mathcal{Q}_n(\mathbb{Z}_p)}.$$

Remarks

- First example of such a fundamental lemma.
- First major step in stabilizing the relative trace formula for unitary Friedberg-Jacquet periods:

$$\mathbf{1}_{\mathcal{Q}_n(\mathbb{Z}_p)} \longleftrightarrow \mathbf{1}_{\mathcal{Q}_a(\mathbb{Z}_p)\times\mathcal{Q}_b(\mathbb{Z}_p)}$$
$$J^{\kappa}(f) \longleftrightarrow SJ(f^{a,b})$$

- Still need to establish existence of smooth transfer (partial results sufficient for some purposes are known.)

Idea of proof

Step 1: Reduce to the tangent space at the $H(F)$-fixed point

$$T_e(\mathcal{Q}_n(F)) \cong \mathrm{End}(V).$$

Then $\mathrm{End}(\Lambda_n) \subset \mathrm{End}(V_n)$, and we prove

$$\mathrm{SO}((X_a, X_b), \mathbf{1}_{\mathrm{End}(\Lambda_a)\oplus\mathrm{End}(\Lambda_b)}) = \Delta(X, (X_a, X_b))\mathrm{O}^{\kappa}(X, \mathbf{1}_{\mathrm{End}(\Lambda_n)}).$$

Step 2: Prove this linearized version follows from a fundamental lemma for a full Hecke algebra on

$$X_n = \mathrm{GL}(V_n)/U(V_n).$$

Idea of proof

Step 3: Using an uncertainty principle and a novel result of J. Xiao (relating endoscopy for unitary Lie algebras to Jacquet-Rallis transfer), prove this follows “from a germ expansion” of a comparison of functions between

$$X_n \text{ and } \mathrm{GL}_n(F).$$

Step 4: Introduce a new comparison of (distinct) RTFs to prove this via globalization argument.

Where do these endoscopic spaces come from?

Fix X and κ, and let's assume that X is **elliptic** (eg: H_X is anisotropic).

$$\begin{array}{ccc} & H^1(F, H_X) & \xrightarrow{\kappa} \mathbb{C}^\times \\ & \uparrow \text{Tate-Nakayama} & \\ & H^{-1}(F, X_*(H_X)) & \\ & \uparrow = & \\ X_*(H_X) \longrightarrow & X_*(H_X)/I_{aug}X_*(H_X) & \end{array}$$

Thus, we obtain

$$\tilde{\kappa} : X_*(H_X) \longrightarrow \mathbb{C}^\times .$$

Root system

Claim

There is a natural coroot system $R^\vee_{\hat{\mathcal{Q}}} \subset X_*(H_X) \cong X^*(\hat{H}_X)$.

(Sakellaridis-Venkatesh, Gaitsgory-Nadler, Knop-Schalke): The spherical variety $\mathcal{Q} = G/H$ has a dual group $\hat{G}_{\mathcal{Q}} = \mathrm{Sp}_{2n}(\mathbb{C})$ which is naturally equipped with a morphism

$$\mathrm{Sp}_{2n}(\mathbb{C}) \longrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \longrightarrow \hat{\mathcal{Q}}(\mathbb{C})$$

There is a natural embedding $\hat{H}_X \subset \hat{\mathcal{Q}}(\mathbb{C})$, and the root system $R_{\hat{\mathcal{Q}}}$ is the root system of this (dual) symmetric space.

Endoscopic root system

Set $R^{\kappa}_{\hat{\mathcal{Q}}} = R_{\hat{\mathcal{Q}}} \cap \ker(\tilde{\kappa})$. Gives a root system, and a diagram

$$\begin{array}{ccccc}
\mathrm{Sp}_{2a}(\mathbb{C}) \times \mathrm{Sp}_{2b}(\mathbb{C}) & \longrightarrow & \mathrm{GL}_{2a}(\mathbb{C}) \times \mathrm{GL}_{2b}(\mathbb{C}) & \longrightarrow & \hat{\mathcal{Q}}^{\kappa}(\mathbb{C}) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Sp}_{2n}(\mathbb{C}) & \longrightarrow & \mathrm{GL}_{2n}(\mathbb{C}) & \longrightarrow & \hat{\mathcal{Q}}(\mathbb{C})
\end{array}$$

Top row is dual to $\mathcal{Q}_a \times \mathcal{Q}_b$.

Question

Does this generalize to other spherical varieties?

THANK YOU!!