Whittaker functions and connections to crystal graphs

Spencer Leslie [1]

[1] Department of Mathematics
Duke University

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Goals

- Study metaplectic Whittaker functions
- Derive generalized Tokuyama-type formulas (esp. G_2)
- Feedback between p-adic Whittaker functions and geometry
  1. (geometry $\Rightarrow$ Whittaker) connect to crystal graphs
  2. (Whittaker $\Rightarrow$ geometry) Reveal new structure on families of MV polytopes.

Tokuyama's formula

- The **Schur polynomial**

$$s_\lambda(x) = \sum_{\text{shape } \lambda} x^T$$

  is the character of the finite dim'l representation $V(\lambda)$ of SL_r of highest weight λ.
- **Tokuyama ('88):** formula for the product of s_λ and a deformation of the Weyl denominator.
- **Brubaker-Bump-Friedberg (2010):** reinterpret as a sum over *Kashiwara crystal graphs*.

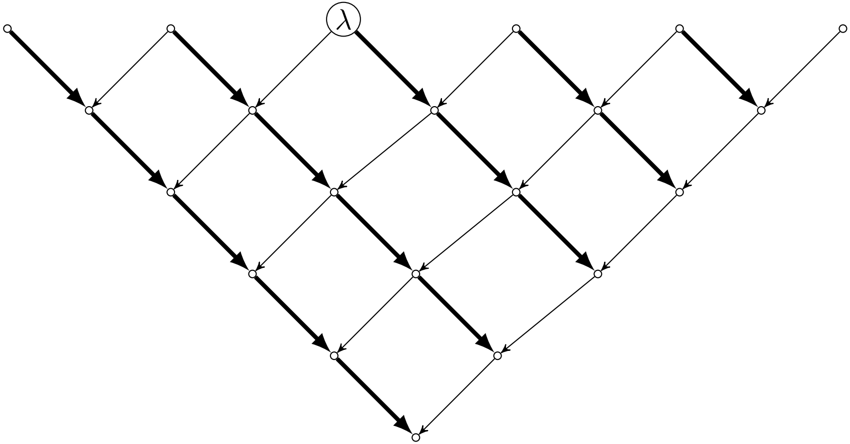
Crystal graphs

Kashiwara's theory of **crystal graphs** gives a combinatorial model for highest weight representations of a semisimple complex Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. For a dominant weight λ,

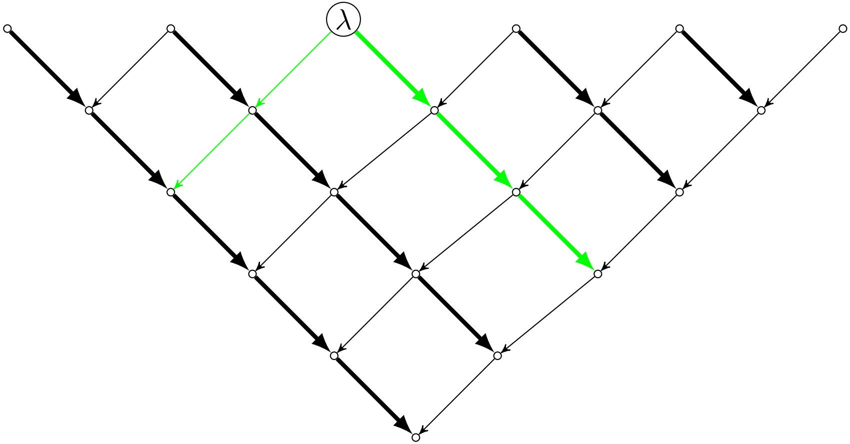
$$V(\lambda) = \text{ finite dim'l representation of highest weight } \lambda$$
$$\mathcal{B}(\lambda) = \text{Kashiwara crystal that parametrizes weight basis}$$

All these finite crystals $\mathcal{B}(\lambda)$ may be given as distinguished subsets of the infinite crystal $\mathcal{B}(-\infty)$.

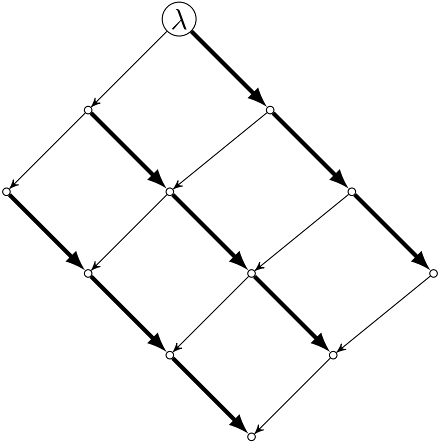
Toy picture: $\mathcal{B}(-\infty)$ for $\Phi = A_1 \times A_1$



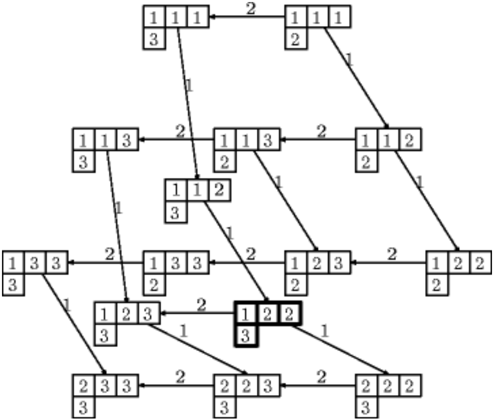
Toy picture: Bounding hyperplanes for $\mathcal{B}(1,2)$



Toy picture: $\mathcal{B}(\lambda)$ for $\lambda = (1, 2)$



Crystal graphs: A_2, $\lambda = 2\omega_1 + \omega_2$



Kashiwara crystal graphs:

$$(\mathcal{B}(\lambda), e_i, f_i, wt, \ldots)$$

here

$$e_i = \text{ raising operator for } \alpha_i,$$

$$f_i = \text{ lowering operator for } \alpha_i,$$

for α_i the simple roots.

Tokuyama's formula

Theorem (Tokuyama)

Let λ be a dominant weight of $\mathrm{SL}_r(\mathbb{C})$, and let $z \in T(\mathbb{C})$. Then

$$\prod_{\alpha \in \Phi^+} \left(1 - q^{-1} z^{\alpha}\right) s_{\lambda}(z) = \sum_{\nu \in \mathcal{B}(\lambda+\rho)} H(\nu) z^{wt(\nu) - w_0(\lambda+\rho)}.$$

Here, $wt : \mathcal{B}(\lambda + \rho) \longrightarrow X^*(T)$ is the weight map of the finite crystal $\mathcal{B}(\lambda + \rho)$.

The Tokuyama function H is defined combinatorially in terms of how ν sits inside $\mathcal{B}(\lambda + \rho)$.

Example: $SL(2)$, $\lambda = 3\omega$, $V(\lambda) = \mathrm{Sym}^3(\mathbb{C}^2)$

For $z = \mathrm{diag}(t, t^{-1})$ for $t \in \mathbb{C}^\times$.

Example of Tokuyama's identity

$$(1 - q^{-1}t^2)\overbrace{(t^3 + t + t^{-1} + t^{-3})}^{s_\lambda(z)}$$
$$= t^{-3} + (1 - q^{-1})t^{-1} + (1 - q^{-1})t$$
$$+ (1 - q^{-1})t^3 + (-q^{-1})t^5$$

Note: $\lambda + \rho = 4\omega$.

Notation

- split simple algebraic group $G = G(\mathbb{Q}_p)$ (for a p a prime).
- Opposite Borels B, B^-,
- unipotent radicals U, U^-,
- split maximal torus $T = B \cap B^-$.

Remark

Much of this generalizes to n-fold covers of G, but unless otherwise stated we assume $n = 1$.

Whittaker functions

Tokuyama's formula and its generalizations are related to Spherical Whittaker functions.

- Fix a generic character $\psi : U^- \longrightarrow \mathbb{C}^\times$, and a dominant coweight λ.
- Let $\tau \in T^\vee(\mathbb{C}) \subset G^\vee(\mathbb{C})$. There is a unique function $\varphi_\tau : G \to \mathbb{C}$, called the a (normalized) spherical function, such that
$$\varphi(bk) = \delta^{1/2}\tau(b)$$
for $k \in G(\mathbb{Z}_p)$, $b \in B$. (Note: $G = B \cdot G(\mathbb{Z}_p)$)

Spherical Whittaker function

We are interested in the integral

$$I_\lambda(\tau) := \int_{U^-} \varphi_\tau(u)\psi\left(p^\lambda u p^{-\lambda}\right) du.$$

Casselman-Shalika

Theorem (Casselman-Shalika)

Fixing $\tau \in T^\vee(\mathbb{C})$ and λ dominant as before,

$$I_\lambda(\tau) = \tau^{w_0\lambda} \prod_{\alpha\in\Phi^+} \left(1 - p^{-1}\tau^\alpha\right) \chi_\lambda(\tau),$$

where χ_λ is the character of the finite dim'l representation $V(\lambda)$ of $\mathfrak{g}^\vee = \mathrm{Lie}(G^\vee)$.

Tokuyama's formula gives an expression of $I_\lambda(\tau)$ as a sum over $\mathcal{B}(\lambda+\rho)$ in type A.

Goal

Generalize this formula for the spherical Whittaker function to other types and covering groups

Motivations/Applications

Spherical Whittaker functions arise when computing Fourier coefficients of Eisenstein series

- Play central role in Langlands-Shahidi and Rankin-Selberg methods of studying automorphic L-functions

When our group G is replaced by a *metaplectic covering group*

$$1 \to \mu_n \to \widetilde{G} \to G \to 1,$$

these coefficients have several applications to number theory:

- eg: moments of L-functions, multiple Dirichlet series, metaplectic automorphic forms
- Standard techniques do not work for covering groups: need new methods for computing Spherical Whittaker functions!

A bit more notation

- Let $w_0 \in W$ be the longest element.
- If $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ is minimal, we call $\underline{i} = (i_1, i_2, \cdots, i_N)$ a reduced word for w_0.
- The choice of $\underline{i}$ orders the positive roots Φ^+:

Example

$G = \mathrm{SL}(3)$ The choice $\underline{i} = (1, 2, 1)$ induces

$$\alpha_2 < \alpha_1 + \alpha_2 < \alpha_1.$$

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- Let $w_0 \in W$ be the longest element.
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Example

$G = \mathrm{SL}(3)$ The choice $\underline{i} = (2, 1, 2)$ induces

$$\alpha_2 > \alpha_1 + \alpha_2 > \alpha_1.$$

Crystal expansion

The most direct connection between Whittaker functions and crystals is geometric.

Theorem (McNamara)

Let $\underline{i} = (i_1, \ldots, i_N)$ be a reduced expression of w_0. There is a decomposition (given by an Iwasawa decomposition)

$$U^- = \bigsqcup_{\mathbf{m} \in \mathcal{B}(-\infty)} C^{\underline{i}}(\mathbf{m}).$$

In particular, the Spherical Whittaker function I_λ has a formal expansion

$$I_\lambda = \sum_{\mathbf{m} \in \mathcal{B}(-\infty)} I_\lambda(\mathbf{m}).$$

Idea behind this decomposition

Let $G = \mathrm{SL}(2)$, $x \in \mathbb{Q}_p$.

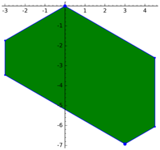
$$\begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & \\ x & 1 \end{pmatrix} & \text{if} \quad x \in \mathbb{Z}_p \\ \begin{pmatrix} x^{-1} & 1 \\ & x \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & x^{-1} \end{pmatrix} & \text{if} \quad x \notin \mathbb{Z}_p. \end{cases}$$

Setting $m = -\min\{val(x), 0\}$, then $U^{-} = \sqcup_{m \geq 0} C(m)$.
For higher rank groups, the choice of $\underline{i}$ orders the positive roots, so we can apply this step root-by-root.
$\#$ of cases = $2^{\#\text{pos. roots}}$.

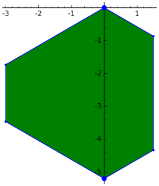
Connection to MV polytopes

- Connects on work of Kamnitzer on (stable) Mirkoviċ-Vilonen (MV) polytopes.
- Moment polytopes of (MV) cycles that play a central role in the geometric Satake equivalence in the geometric Langlands program.
- In bijection with $\mathcal{B}(-\infty)$, allowing connection with crystal graphs. (**requires transcendental techniques!**)
- The Iwasawa algorithm recovers the ***i*-Lusztig data** $\mathbf{m} = (m_1, \ldots, m_N) \in \mathbb{Z}^N_{\geq 0}$ of an MV polytope

MV polytopes for $\Phi = A_2$, $\underline{i} = (1, 2, 1)$



$\mathbf{m} = (1, 2, 3)$



$\mathbf{m} = (1, 2, 1)$

Our choice of $\underline{i}$ implies that the Lusztig data encodes the edge lengths on the *right-ward* path.

Crystal expansion

Theorem (McNamara)

Let $\underline{i} = (i_1, \ldots, i_N)$ be a reduced expression of w_0. There is a formal expansion of the Spherical Whittaker function I_λ has a sum

$$I_\lambda = \sum_{\mathbf{m} \in MV} I_\lambda(\mathbf{m}),$$

where MV is the set of MV polytopes of type G.

For applications, both to number theory and representation theory, need to compute these terms and reduce to a finite sum.

Expectations

$G = \mathrm{SL}_{r+1}$, **best possible $\underline{i}$:** recovers Tokuyama's formula (as well as metaplecitc version of Brubaker-Bump-Friedberg)

Conjecture (McNamara,'09)

For a general group G and reduced word $\underline{i}$,

$$I_\lambda(\mathbf{m}) = 0 \text{ unless } \mathbf{m} \in \mathcal{B}(\lambda + \rho).$$

- **FALSE:** Already counter-examples exist with $G = \mathrm{GL}(4)$.
- The reason why shines a spotlight on an interesting family of MV-polytopes.

Resonant MV-polytopes

Fix a long word $\underline{i} = (i_1, i_2, \ldots, i_N)$. The set of all (stable) MV polytopes MV is in bijection

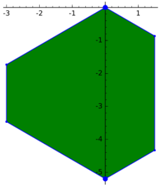
$$MV \longrightarrow \mathcal{B}(-\infty),$$

by associating to M its $\underline{i}$-Lusztig data $\mathbf{m} = (m_1, \ldots, m_N)$.

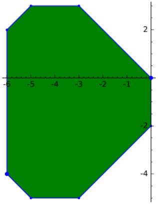
Rough definition

Let M have $\underline{i}$-Lusztig data $\mathbf{m}$. We say M has a **resonance** if there are two indices $k \neq l$ with $i_k = i_l$ and $m_k = m_l$, where $\underline{i} = (i_1, \ldots, i_N)$.

Examples of resonant MV polytopes



Type A_2, $\underline{i} = (1, 2, 1)$
$\mathbf{m} = (1, 2, 1)$



Type B_2, $\underline{i} = (1, 2, 1, 2)$:
$\mathbf{m} = (1, 2, 3, 2)$

Our choice of $\underline{i}$ implies that the Lusztig data encodes the edge lengths on the *right-ward* path.
These polytopes have additional symmetries, but do not seem to be singled out in the literature.

Computing the algorithm

The decomposition produces certain coordinates $\{(t_\alpha, w_\alpha)\}_{\alpha\in\Phi^+}$ whose valuations encode the Lusztig data. $[m_\alpha = -\mathrm{val}(w_\alpha)]$

- Study the MV-integrals

$$I_\lambda(\mathbf{m}) = \int_{C^{\underline{i}}(\mathbf{m})} \varphi_\tau(u)\psi_\lambda(u)du$$

Obstacle

We need to compute $\psi_\lambda(u)$ in terms of the $\underline{i}$-Lusztig data.

- **Idea:** Take advantage connection with MV polytopes to use algebro-geometric work of Berenstein-Zelevinsky-Fomin on **total positivity**.

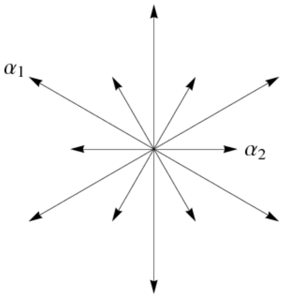
Connecting to total positivity (L)

1. Write $\psi_\lambda(u) = \psi(\mathfrak{s}_1(u) + \cdots \mathfrak{s}_r(u))$.
2. The algorithm implies $s_i(u) = h_i(t_\alpha, w_\alpha)$ is a Laurent polynomial with $\mathbb{Z}$-coefficients
3. **Key step:** interpret in terms of certain birational map used in studying total positivity in double Bruhat cells.

Corollary

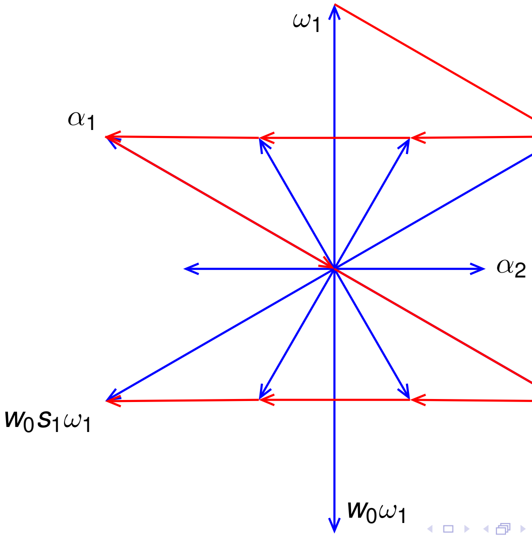
You can explicitly compute $h_i(t_\alpha, w_\alpha)$ as a sum of paths through the weight lattice: these paths are called $\underline{i}$-trails.

Exceptional group G_2

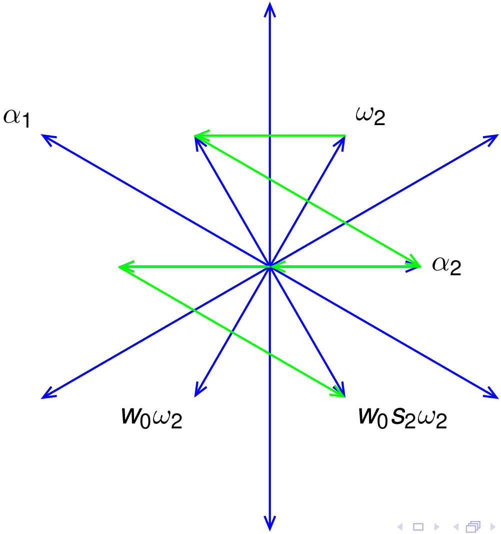


- We choose the reduced decomposition $w_0 = s_2 s_1 s_2 s_1 s_2 s_1$.
- $\alpha_2 < \alpha_1 + 3\alpha_2 < \alpha_1 + 2\alpha_2 < 2\alpha_1 + 3\alpha_2 < \alpha_1 + \alpha_2 < \alpha_1$
- ω_1, ω_2 the fundamental weights

i-trails $\pi : \omega_1 \to w_0 s_1 \omega_1$, $i = (2, 1, 2, 1, 2, 1)$



i-trails for $h_2(t_\alpha, w_\alpha)$, $i = (2, 1, 2, 1, 2, 1)$



The polynomials $h_i(t_\alpha, w_\alpha)$ for G_2

Using the formula, we find that

$$h_1(t_\alpha, w_\alpha) = t_6 = X_6$$

and

$$h_2(t_\alpha, w_\alpha) = \frac{t_1 w_2 w_3}{w_5 w_6} + \frac{t_2 w_3^2}{w_5 w_6} - t_5 + 2\frac{t_3 w_4}{w_6} - \frac{t_3^2 w_4}{w_5 w_6} - \frac{t_4 w_5}{w_6}.$$

Consequences

1. This formula gives a (new?) simple way of writing down the bounding hyperplanes for $\mathcal{B}(\lambda)$.
2. Not hard to show that $I_\lambda(\mathbf{m}) = 0$ for *generic* $\mathbf{m} \notin \mathcal{B}(\lambda + \rho)$.
3. The remaining cases are identified with a certain set of **resonant polytopes**, which need to be computed.

Remainder of Talk: application to G_2

Focus on the case of G_2 to illustrate the geometry of resonant polytopes and derive a **new Tokuyama-type formula** for highest weight characters of type G_2.

MV-integrals for G_2

Fix $\mathbf{m} \in \mathcal{B}(-\infty)$, and fix a dominant weight $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2$.

MV-integral

$$I_\lambda(\mathbf{m}) = \prod_{\alpha\in\Phi^+} (q^{-1}x_\alpha)^{m_\alpha} \int_{C^i(\mathbf{m})} \psi\left(\varpi^{\lambda_1} t_6 + \varpi^{\lambda_2} h_2(t_\alpha, w_\alpha)\right) dt.$$

$$h_2(t_\alpha, w_\alpha) = \frac{t_1 w_2 w_3}{w_5 w_6} + \frac{t_2 w_3^2}{w_5 w_6} - t_5 + 2\frac{t_3 w_4}{w_6} - \frac{t_3^2 w_4}{w_5 w_6} - \frac{t_4 w_5}{w_6}.$$

Computing the MV-integrals for G_2

Evaluation of integrals (L)

Let $I_\lambda(\mathbf{m})$ be the MV-integral associated to **m**.

- If $\mathbf{m} \in \mathcal{B}(\lambda+\rho)$, $I_\lambda(\mathbf{m})$ is the standard contribution.
- If $\mathbf{m} \notin \mathcal{B}(\lambda+\rho)$ then $I_\lambda(\mathbf{m}) = 0$ unless

$$m_3 = m_5, \ \ m_4 > 0, \text{ and } \mathrm{val}(X_4) = \lambda_2 + 2k.$$

In all cases, the integrals a simple polynomials in p^{-1}.

- Infinitely many resonant MV cycles with non-zero contribution.

Resonant MV-polytopes for G_2

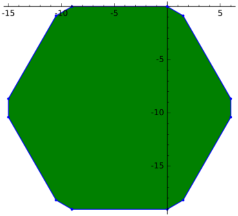
Definition

An Mikovìc-Vilonen polytope M for G_2 is **resonant** for the long word $\underline{i} = (2, 1, 2, 1, 2, 1)$ if $m_3 = m_5$, where M has $\underline{i}$-Lusztig data

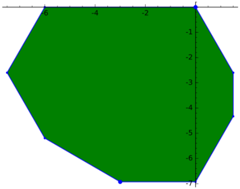
$$\mathbf{m} = (m_1, m_2, m_3, m_4, m_5, m_6).$$

We say it is **totally resonant** if $m_2 = m_6$ as well.

Examples of Resonant MV polytopes, $\Phi = G_2$



$\mathbf{m} = (3, 1, 3, 1, 3, 1)$



$\mathbf{m} = (1, 0, 1, 1, 1, 0)$

Again, our choice of $\underline{i}$ has the Lusztig data encoding edge lengths on the “right-ward” path.

Resonance families

Keep our dominant weight λ and fix a weight μ.

- In general, no additional structure on the set of (stable) MV-polytopes of weight μ.

Theorem (L)

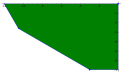
For a fixed weight μ, the set $RMV(\mu)$ of resonant MV-polytopes of weight μ may be equipped with operations

$$e_1, e_2, f_1, f_2 : RMV(\mu) \longrightarrow RMV(\mu) \cup \{0\},$$

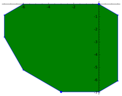
which decompose the set into a disjoint union of (truncated) Kashiwara crystal graphs of type $A_1 \times A_1$.

New combinatorial representation theoretic structure on resonant polytopes.

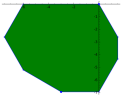
Example: Resonance family of weight $(8, 5)$



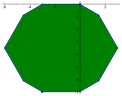
$$\mathbf{m} = (1, 0, 0, 4, 0, 0)$$



$$\mathbf{m} = (1, 1, 0, 3, 0, 1)$$



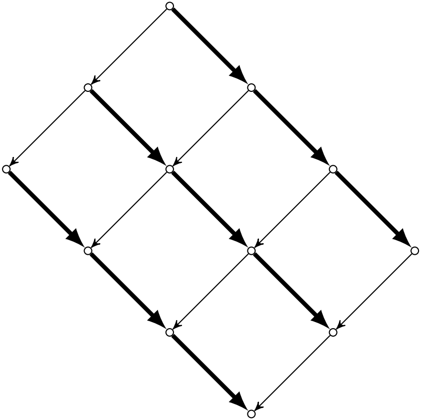
$$\mathbf{m} = (1, 0, 1, 1, 1, 0)$$



$$\mathbf{m} = (1, 1, 1, 0, 1, 1)$$

Families outside of $\mathcal{B}(\lambda + \rho)$

- Most Resonance families do not meet $\mathcal{B}(\lambda + \rho)$,
- These are full crystal graphs.



Sums over Resonance Families

These families are detected by the sums in the spherical Whittaker function

Theorem (L)

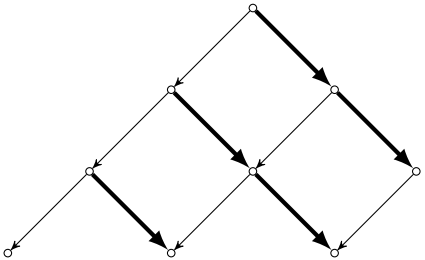
Let RF be the resonance family such that $RF \cap \mathcal{B}(\lambda + \rho) = \emptyset$. Then

$$\sum_{\mathbf{m} \in RF} I_\lambda(\mathbf{m}) = 0.$$

This reduces I_λ to a sum over finitely many bases elements, but more than $\mathcal{B}(\lambda + \rho)$.

λ- Relevant Resonance Families

- In a **finite number of cases**, a resonance family with intersect $\mathcal{B}(\lambda+\rho)$.
- The crystal graph is truncated at the wall of $\mathcal{B}(\lambda+\rho)$
- The sum cannot be cancelled$\Rightarrow$ must incorpertate these terms into the sum.



Tokuyama-esque formula for G_2

Theorem (L)

For λ dominant weight and $\tau \in T^\vee(\mathbb{C}) \subset G_2(\mathbb{C})$, then

$$\prod_{\alpha \in \Phi^+} \left(1 - p^{-1}\tau^\alpha\right) \chi_\lambda(\tau)$$

$$= \sum_{\nu \in \mathcal{B}(\lambda+\rho)} H^*_{std}(\nu)\tau^{wt(\nu)-w_0(\rho)} + \sum_{\substack{RF(\mathbf{m}) \\ \lambda\text{-relevant}}} Err_\lambda(q^{-1}, \mathbf{m}),$$

for an explicit family of “geometric error” polynomials $Err_\lambda(q^{-1}, \mathbf{m})$.

It is expected that this “geometric error term” is related to reduction multiplicities $G_2 \rightsquigarrow SL_2$.

Conclusion

- (geometry $\Rightarrow$ Whittaker:) connection to crystal graphs (uses transcendental methods)
- (Whittaker $\Rightarrow$ geometry:) p-adic techniques reveal hidden structure on MV polytopes.