Arithmetic Wavefront Set, Local Descent, and Local Gan-Gross-Prasad Conjecture

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Brief Introduction to Wavefront Set

F a local field: $\mathbb{C}$, $\mathbb{R}$, or finite extension of $\mathbb{Q}_p$.

G reductive algebraic group defined over F: $G = G(F)$

$\Pi_F(G)$ the set of equiv. classes of irreducible admissible representations of G, which are of Casselman-Wallach type if F is archimedean.

The admissibility of $\pi \in \Pi_F(G)$ shows that for $f \in_c^\infty (G)$,

$$\pi(f) := \int_G f(g)\pi(g)dg$$

is of trace class, and hence $\Theta_\pi(f) := \mathrm{tr}_\pi(f)$ is well-defined.

Harish-Chandra Theorem: The trace character Θ_π determines $\pi \in \Pi_F(G)$ uniquely up to equivalence.

Brief Introduction to Wavefront Set

Harish-Chandra Regularity Theorem shows that Θ_π is locally L^1-function on G (real analytic over the G^r if $F = \mathbb{R}$ or $\mathbb{C}$).

The asymptotic behavior of Θ_π at the identity is expressed by means of the nilpotent orbits of $\mathfrak{g}$, the Lie algebra of G.

$\mathcal{N}_F(\mathfrak{g})$ is the set of F-rational nilpotent elements in $\mathfrak{g}(F)$.

$\mathcal{N}_F(\mathfrak{g})_\circ$ is the set of all F-rational $\mathrm{Ad}(G)$-orbits in $\mathcal{N}_F(\mathfrak{g})$.

$\mathcal{N}_F(\mathfrak{g})_\circ^{\mathrm{st}}$ is the set of F-stable orbits from $\mathcal{N}_F(\mathfrak{g})_\circ$.

Classical Theorem: The set $\mathcal{N}_F(\mathfrak{g})_\circ^{\mathrm{st}}$ is finite and is parameterized by combinatorical data (partitions or Bala-Carter data).

The classification of the set $\mathcal{N}_F(\mathfrak{g})_\circ$ is also known via various methods by many people.

Distribution Character and Wavefront Set

R. Howe (1974) and Harish-Chandra (1978): When F is p-adic, the distribution character Θ_π has an expansion near the identity:

$$\Theta_\pi = \sum_{\mathcal{O}\in\mathcal{N}_F(\mathfrak{g})_\circ} c_{\mathcal{O}} \cdot \widehat{\mu}_{\mathcal{O}}, \quad c_{\mathcal{O}} \in \mathbb{C},$$

where $\widehat{\mu}_{\mathcal{O}}$ is the Fourier transform of the measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$.

Around 1980, D. Barbasch and D. Vogan developed a similar asymptotic expansion of Θ_π when F is archimedean.

It is an interesting **question** to understand the impacts of the following set

$$\mathcal{C}_{\mathrm{tr}}(\pi) := \{\mathcal{O} \in \mathcal{N}_F(\mathfrak{g})_\circ \mid c_{\mathcal{O}} \neq 0\}$$

towards the structure of π.

Brief Introduction to Wavefront Set

If θ is a distribution on a manifold X, the **Wavefront Set** $\mathrm{WF}(\theta)$ was introduced by L. Hörmander around 1970, which is a closed subset in the cotangent bundle $T^*(X)$ of X.

For $\pi \in \Pi_F(G)$ when $X = G$ is a Lie group, R. Howe defined, around 1980, the **Wavefront Set** $\mathrm{WF}_H(\pi)$, which is the projection of $\mathrm{WF}(\Theta_\pi)$ to the dual $\mathfrak{g}^*$ of $\mathfrak{g}$.

With the identification $\mathfrak{g} \cong \mathfrak{g}^*$, R. Howe Showed that

$$\mathrm{WF}_H(\pi) \subset \mathcal{N}_F(\mathfrak{g}).$$

Since Θ_π is adjoint invariant, $\mathrm{WF}_H(\pi)$ is adjoint-stable, and hence

$$\mathrm{WF}_H(\pi) \subset \mathcal{N}_F(\mathfrak{g})_\circ.$$

Brief Introduction to Wavefront Set

Howe suggested in his 1980 paper to understand the relation of the two subsets

$$\mathcal{C}_{\mathrm{tr}}(\pi) \quad \text{and} \quad \mathrm{WF}_H(\pi) \subset \mathcal{N}_F(\mathfrak{g})_\circ$$

for $\pi \in \Pi_F(G)$.

Call $\mathrm{WF}_H(\pi)$, the **Geometric Wavefront Set** of π.

Call $\mathrm{WF}_{\mathrm{tr}}(\pi) := \mathcal{C}_{\mathrm{tr}}(\pi)$, the **Analytic Wavefront Set** of π.

Question: For any $\pi \in \Pi_F(G)$, with G reductive,

$$\mathrm{WF}_H(\pi) = \mathrm{WF}_{\mathrm{tr}}(\pi)? \quad \text{or} \quad \mathrm{WF}_H(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}?$$

Brief Introduction to Wavefront Set

I refer to D. Vogan's Takagi Lecture 2016 for more detailed discussion of the significance of wavefront sets and their relatives.

M. Raibaut (2019): Wavefront Sets and Motivic Integrstion.

N. Kawanaka (1987): **Wavefront Set** $\mathrm{WF}_K(\pi)$ (using generalized Gelfand-Graev representations) for p-adic groups.

Mœglin and Waldspurger (1987): **Wavefront Set** $\mathrm{WF}_{MW}(\pi)$ (using degenerate Whittaker models) for p-adic groups.

We call

$$\mathrm{WF}_{\mathrm{wm}}(\pi) := \mathrm{WF}_K(\pi) = \mathrm{WF}_{MW}(\pi)$$

the **Algebraic Wavefront Set** of $\pi \in \Pi_F(G)$ when F is p-adic.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ for $\pi \in \Pi_F(G)$ can be defined when F is archimedean.

Degenerate Whittaker Model and Wavefront Set

For $X \in \mathcal{N}_F(\mathfrak{g})$, there is an $\mathfrak{sl}_2$-triple $\{X, \hbar, Y\}$.

Via $\mathrm{ad}(\hbar)$, $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\ \mathfrak{g}_i^{\hbar}$ with $\mathfrak{g}_i^{\hbar} := \{x \in \mathfrak{g} \colon \mathrm{ad}(\hbar)(x) = ix\}$.

Denote by $\mathfrak{u} = \oplus_{i\leq -1}\ \mathfrak{g}_i^{\hbar}$, $\mathfrak{p} = \oplus_{i\leq 0}\ \mathfrak{g}_i^{\hbar}$, and $\mathfrak{m} = \mathfrak{g}_0^{\hbar}$.

Denote by $U_X = U = \exp(\mathfrak{u})$, and similar P and M.

$P = M \ltimes U$ is a parabolic of G, depending on $\mathcal{O}_X^{\mathrm{st}}$ up to conjugate.

Take $\mathfrak{u}_{X,2} = \oplus_{i\leq -2}\ \mathfrak{g}_i^{\hbar}$ and $U_{X,2} = \exp(\mathfrak{u}_{X,2})$.

Fix a character $\psi_0 \neq 1$ of F and an inv. nondeg. bilinear form κ.

$$\psi_X(\exp(A)) = \psi_0(\kappa(X, A)) \qquad \text{for } A \in \mathfrak{u}_{X,2}$$

defines a character of $U_{X,2}$.

Degenerate Whittaker Model and Wavefront Set

If $\mathfrak{g}_{-1} \neq 0$, $\kappa_{-1}(A, B) := \kappa(\mathrm{ad}(X)A, B)$ for $A, B \in \mathfrak{g}_{-1}$ defines a symplectic form.

$\mathcal{H}_X = \mathfrak{g}_{-1} \times F$ is a Heisenberg group with a surjective morphism: $\alpha_X \colon U_X \mapsto \mathcal{H}_X$ given by

$$\alpha_X(\exp(A)\exp(Z)) = (A, \kappa(X, Z)) \qquad \text{for } A \in \mathfrak{g}_{-1}, Z \in \mathfrak{u}_{X,2}.$$

ψ_X factors through $U_{X,2}$ and defines a central character of $\mathcal{H}_X$.

The oscillator representation $(\omega_{\psi_X}, V_{\psi_X})$ of $\mathcal{H}_X$ is also that of U_X.

Mœglin-Waldspurger (1987); Gomez-Gourevitch-Sahi (2017), for $(\pi, V_\pi) \in \Pi_F(G)$ define **degenerate Whittaker modules** of π:

$$\mathcal{J}_X(\pi) = V_\pi \widehat{\otimes} V_{\psi_X}^\vee \ / \ \overline{\{\pi \otimes \omega_{\psi_X}^\vee(u)v - v \mid u \in U_X, v \in V_\pi \widehat{\otimes} V_{\psi_X}^\vee\}},$$

which depends only on the F-rational nilpotent orbit of X.

Algebraic Wavefront Set

Define the **Algebraic Wavefront Set** of π by

$$\mathrm{WF}_{\mathrm{wm}}(\pi) := \{\mathcal{O} \in \mathcal{N}_F(\mathfrak{g})_\circ \mid \mathcal{J}_X(\pi) \neq 0, \text{ for some } X \in \mathcal{O}\}.$$

Question: For any $\pi \in \Pi_F(G)$,

$$\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max} = \mathrm{WF}_H(\pi)^{\max}?$$

Mœglin-Waldspurger (1987) show: For F p-adic,

$$\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}, \quad \text{for} \quad \pi \in \Pi_F(G).$$

However, in the archimedean case, this equality is not fully understood, except several special cases considered in Vogan (1978, 2016), H. Matumoto (1987, 1990, 1992), H. Yamashita (2001), D. Gourevitch-S. Sahi (2015), and N. Li (2022)

Gomez-Gourevitch-Sahi (2017): G, $F = \mathbb{C}$, or $G = \mathrm{GL}_n$, $F = \mathbb{R}$.

Computing Wavefront Set

Motivated by the understanding of the unitary dual via the orbit method, one computes $\mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$ for $\pi \in \Pi_F(G)$.

Adams-Vogan (2021) provides an (atlas) algorithm to compute the wavefront sets or associated varieties for $\pi \in \Pi_{\mathbb{R}}(G)$ by using the Kazhdan-Lusztig theory, the Knapp-Zuckerman theory, the equivariant K-theory, and the work of W. Schmid-K. Vilonen.

If F is p-adic and π unipotent representations in the sense of Arthur (1984), Waldspurger (2018,2019,2020) determines the wavefront sets for SO_{2n+1} and Ciubotaru, Mason-Brown, and Okada (2021,2022) determines that for general reductive groups, but for Iwahori-spherical π, using the Lusztig classification of unipotent representations (1995, 2002).

D. Jiang-L. Zhang (2018) aimed to understand (and computing) $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ by using local descent.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ and Local Descent: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

G is a split classical group, say, $G = \mathrm{SO}_{2n+1}(F)$.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ consists of orthogonal partitions $\underline{p} = [p_1 p_2 \cdots p_r]$ of $2n+1$, with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$.

Question: For a given $\pi \in \Pi_F(G)$, what is the largest part p_1 such that the partition

$$\underline{p} = [p_1 \cdot 1^*]$$

occurs in $\mathrm{WF}_{\mathrm{wm}}(\pi)$?

For $G = \mathrm{SO}_{2n+1}(F)$, such a partition must be of the form

$$\underline{p}_\ell := [(2\ell+1)1^{2(n-\ell)}]$$

with $\ell = 0, 1, \cdots, n$.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ and Local Descent: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

When $\underline{p}_\ell$ corresponds to $\mathcal{O}_\ell$, for $X \in \mathcal{O}_\ell$, one may choose

$$U_X = U_{\ell,X} = \{u = \begin{pmatrix} z & x & w \\ 0 & \mathrm{I}_{2(n-\ell)+1} & x' \\ 0 & 0 & z' \end{pmatrix}\}, \quad \mathfrak{g}_{-1} = 0,$$

with z a upper-triangular unipotent in GL_ℓ. The character $\psi_X = \psi_{\ell,X}$ can be chosen to be

$$\psi_{\ell,X}(n) = \psi(z_{12} + \cdots + z_{\ell-1,\ell})\psi(< x_\ell, \alpha_X >)$$

with x_ℓ the last low of x, which is a vector in $F^{2(n-\ell)+1}$, and α_X is an anisotropic vector $F^{2(n-\ell)+1}$, corresponding $X \in \mathcal{O}_\ell$

$M_{\ell,X} = \mathrm{Stab}_{\mathrm{SO}_{2(n-\ell)+1}}(\alpha_X)^\circ$ is an F-rational form of $\mathrm{SO}_{2(n-\ell)}$.

For $\pi \in \Pi_F(G)$, $\mathcal{J}_X(V_\pi)$ is a representation of $M_{\ell,X}$.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ and Local Descent: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

Multiplicity One Theorem: For any $\sigma \in \Pi_F(M_{\ell,X})$, we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_{M_{\ell,X}}(\mathcal{J}_X(V_\pi)\widehat{\otimes}\sigma^\vee, 1) \leq 1.$$

When F is p-adic, it is proved by A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann (2010) for $\ell = 0$.

For general ℓ, it is proved by W. T.Gan, B. Gross, and D. Prasad (2012), using Frobenius reciprocity law and the case $\ell = 0$.

When F is archimedean, it is proved by B. Sun and C. Zhu (2012) for $\ell = 0$.

For general ℓ, it is proved by Jiang-Sun-Zhu (2010) by using the case $\ell = 0$ and explicit construction of certain intertwining operators.

Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

$$
\begin{array}{ccccc}
 & & & \mathcal{J}_{X_0}(V_\pi) & \mathrm{SO}_{2n} \\
 & & \vdots & \vdots & \vdots \\
 & & & \mathcal{J}_{X_{\ell-1}}(V_\pi)) & \mathrm{SO}_{2n-2\ell+2} \\
 & & \nearrow & & \\
\mathrm{SO}_{2n+1} & \pi & \rightarrow & \mathcal{J}_{X_\ell}(V_\pi)) & \mathrm{SO}_{2n-2\ell} \\
 & & \searrow & & \\
 & & & \mathcal{J}_{X_{\ell+1}}(V_\pi)) & \mathrm{SO}_{2n-2\ell-2} \\
 & & \vdots & \vdots & \vdots \\
 & & & \mathcal{J}_{X_n}(V_\pi) & \mathrm{SO}_0
\end{array}
$$

Local Descent: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

For $\pi \in \Pi_F(G)$, there is an ℓ_0 s.t. $\mathcal{J}_{X_0}(V_\pi)) \neq 0$ for some $X_0 \in \mathcal{O}_{\ell_0}$, but $\mathcal{J}_X(V_\pi)) = 0$ for any $\ell_0 < \ell \leq n$; any $X \in \mathcal{O}_\ell$.

This integer $\ell_0 = \ell_0(\pi)$ is called the **first occurrence index** of π.

Write $\mathcal{D}_{\ell_0, X_0}(V_\pi) = \mathcal{J}_{X_0}(V_\pi)$ and call it the **local descent** of π.

Jiang-Zhang (2018): If $\pi \in \Pi_F(G)$ is tempered, then $\ell_0 = \ell_0(\pi)$ can be determined by the endoscopic classification data of π as given by J. Arthur 2013.

Conjecture (JZ2018): For any $\pi \in \Pi_F(G)$ with generic L-parameter, if p_1 is the largest part in $\underline{p} = [p_1 p_2 \cdots p_r]$ in $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$, with $p_1 \geq p_2 \geq \cdots p_r > 0$, then

$$p_1 = 2\ell_0 + 1,$$

with $\ell_0 = \ell_0(\pi)$.

Local Descent: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

Theorem [Discreteness of Local Descent] (JZ2018): Let G be SO_{2n+1} or SO_{2n}. If $\pi \in \Pi_F(G)$ has a generic L-parameter, then the **local descent** $\mathcal{D}_{\ell_0,X_0}(V_\pi)$ has the following properties:

1. $\mathcal{D}_{\ell_0,X_0}(V_\pi) = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_m \oplus \cdots$ is a multiplicity free direct sum,
2. σ_j are irreducible square-integrable representations of M_{ℓ_0,X_0},
3. all its irreducible summands σ_j belong to different Bernstein components.

Those irreducible summands can be understood by means of the local Langlands conjecture and the local Gan-Gross-Prasad conjecture for classical groups.

Local Langlands Conjecture

Local Langlands Conjecture: Any $\pi \in \Pi_F(G)$ can be written as

$$\pi = \pi(\phi, \chi)$$

with a L-parameter

$$\phi\colon \mathcal{W}_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G$$

and a χ, which is a character of the component group $\mathcal{S}_\phi$ of the centralizer S_ϕ.

If G^* is F-quasisplit and G is a pure inner form of G^*. then the local Vogan packet is defined by

$$\Pi_\phi[G^*] := \cup_G \Pi_\phi(G)$$

where $\Pi_\phi(G)$ is the local L-packet of G associated with ϕ, which is the empty set if ϕ is not compatible with G.

Local Langlands Conjecture

Local Langlands Conjecture: Given $\phi \in \Phi_F(G^*)$, there is a bijection

$$\Pi_\phi[G^*] \longleftrightarrow \widehat{\mathcal{S}_\phi}$$

with the property that for any $\pi \in \Pi_\phi[G^*]$, there exists a unique $\chi \in \widehat{\mathcal{S}_\phi}$, s.t. $\pi = \pi(\phi, \chi)$, up to a choice of Whittaker normalization and a choice of the rationality of the local Langlands correspondence.

This version of the **Local Langlands Conjecture** for classical groups over p-adic local fields has been established through the endoscopic classification for classical groups by J. Arthur, and the work of C. Mœglin, and others.

The local Langlands conjecture for reductive groups over archimedean local fields follows from the Langlands classification theory.

Local Gan-Gross-Prasad Conjecture

G_n^*: F-quasisplit SO_{2n+1} and G_n: pure inner F-form of G_n^*

For an odd integer ℓ with $0 < \ell \leq n$, $H_{n-\ell}^*$: F-quasisplit $\mathrm{SO}_{2n-2\ell}$.

$\Phi_{\mathrm{gen}}(G_n^*)$: generic local L-parameters for G_n^*

For any $0 \leq \ell \leq n$, the Hom-space

$$\dim_{\mathbb{C}} \mathrm{Hom}_{M_{\ell,X}}(\mathcal{J}_X(V_\pi)\widehat{\otimes}\sigma^\vee, 1) = 1$$

where $M_{\ell,X}$ is a pure inner form of $H_{n-\ell}^*$, for some $\pi = \pi(\varphi, \chi)$ and $\sigma = \sigma(\phi, \chi')$ with $\varphi \in \Phi_{\mathrm{gen}}(G_n^*)$ and $\phi \in \Phi_{\mathrm{gen}}(H_{n-\ell}^*)$; and $\chi \in \widehat{\mathcal{S}_\varphi}$ and $\chi' \in \widehat{\mathcal{S}_\phi}$ if and only if the pair of the characters (χ, χ') can be computed explicitly by means of the local root numbers, i.e.

$$\chi = \chi_{\varphi,\phi} \quad \text{and} \quad \chi' = \chi_{\phi,\varphi},$$

where the definition of $(\chi_{\varphi,\phi}, \chi_{\phi,\varphi})$ is part of the local GGP.

Local Descent: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

Note that the local GGP conjecture for classical groups is now a theorem of Waldspurger, Mœglin, Beuzart-Plessis, H. Atobe, Gan-Ichino, H. Xue, Z. Luo and C. Chen.

By using the local GGP conjecture and long explicit computation of local root numbers, we obtain

Theorem [Parameterization of Local Descent] (JZ2018): Let G be SO_{2n+1} or SO_{2n}. If $\pi = \pi(\varphi, \chi) \in \Pi_\varphi[G_n^*]$ with $\varphi \in \Phi_{\mathrm{gen}}(G_n^*)$ and $\chi \in \widehat{\mathcal{S}_\varphi}$, then the irreducible square-integrable summands σ_j occurring in the **local descent** $\mathcal{D}_{\ell_0,X_0}(V_\pi)$:

$$\mathcal{D}_{\ell_0,X_0}(V_\pi) = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_m \oplus \cdots$$

has an explicitly given arithmetic data (ϕ_j, χ_j) with $\phi_j \in \Phi_{\mathrm{gen}}(H^*_{n-\ell_0})$ and $\chi_j \in \widehat{\mathcal{S}_{\phi_j}}$, i.e.

$$\chi = \chi^z_{\varphi,\phi_j} \quad \text{and} \quad \chi_j = \chi^z_{\phi_j,\varphi} \quad \text{for some} \quad z \in F^\times.$$

Descent of Enhanced L-parameters

From **[Jiang-Zhang 2018]**, we obtain the following diagram:

$$\begin{array}{ccccc} & \Pi_\varphi[G_n^*] & & \widehat{\mathcal{S}_\varphi} \\ G_n & \pi & \longleftrightarrow & (\varphi, \chi) \\ & \downarrow & & \downarrow \\ H_{n-\ell} & \{\sigma_j\} & \longleftrightarrow & \{(\phi_j, \chi_j)\} \end{array}$$

We start with $\pi \in \Pi_\varphi[G_n^*]$. By the local descent at the first occurrence index, we obtain that all σ are square-integrable.

Hence we may repeat the local descent from σ_j to get the complete "spectrum" of π similar to the Bernstein-Zelevinsky derivatives for $G = \mathrm{GL}_n$.

Descent of Enhanced L-parameters

We now discuss a joint work with D. Liu and L. Zhang (JLZ2022).

With a fixed Whittaker datum, for any $\pi \in \Pi_\varphi[G_n^*]$, there exists a unique $\chi \in \widehat{\mathcal{S}_\varphi}$, s.t. $\pi = \pi(\varphi, \chi)$.

For $(\varphi, \phi) \in \Phi_{\mathrm{gen}}(G_n^* \times H_{n-\ell}^*)$, the **local GGP conjecture** computes $(\chi_{\varphi,\phi}^z, \chi_{\phi,\varphi}^z)$ of $\mathcal{S}_\varphi \times \mathcal{S}_\phi$, for some $z \in \mathcal{Z} := F^\times/F^{\times,2}$.

For $0 < \ell \leq n$, $\varphi \in \Phi_{\mathrm{gen}}(G_n^*)$ and $\chi \in \widehat{\mathcal{S}_\varphi}$, the ℓ**-th descent along** z of (φ, χ) is defined to be a set $\mathfrak{D}_\ell^z(\varphi, \chi)$ that consists of the contragredients $(\widehat{\phi}, \widehat{\chi'})$ of (ϕ, χ') with $\phi \in \Phi_{\mathrm{gen}}(H_{n-\ell}^*)$, $\chi' \in \widehat{\mathcal{S}_\phi}$, and

$$(\chi_{\varphi,\phi}^z, \chi_{\phi,\varphi}^z) = (\chi, \chi')$$

for some $z \in \mathcal{Z}$, and other compatibility under the local GGP conjecture.

Descent of Enhanced L-parameters

The ℓ**-th descent** of (ϕ, χ) is defined to be

$$\mathfrak{D}_\ell(\varphi, \chi) := \cup_{z \in \mathcal{Z}} \mathfrak{D}_\ell^z(\varphi, \chi).$$

The **1st occurrence index** for (ϕ, χ) is defined to be

$$\ell_0 = \mathfrak{l}_0(\phi, \chi) := \max\{0 < \ell \leq n \mid \mathfrak{D}_\ell(\varphi, \chi) \neq \emptyset\}.$$

Theorem [JLZ22] (Discreteness of First Descent): For any enhanced local L-parameter (φ, χ) of G_n^*, the descent at ℓ_0, $\mathfrak{D}_{\ell_0}(\varphi, \chi)$ consists only of discrete enhanced local L-parameters.

This discreteness makes it possible to take consecutive descents of any given enhanced local L-parameters (φ, χ) of G_n^* with $\varphi \in \Phi_{\mathrm{gen}}(G_n^*)$ and $\chi \in \widehat{\mathcal{S}_\varphi}$.

Arithmetic Wavefront Set of (φ, χ)

The **consecutive descents** of (φ, χ) is given by the following diagram:

$$
\begin{array}{llllll}
(\varphi, \chi) & \longrightarrow & \cup_{\ell_1} \mathfrak{D}_{\ell_1}(\varphi, \chi) & & & \\
& & (\varphi_1, \chi_1) & \longrightarrow & \cup_{(\varphi_1, \chi_1)} \cup_{\ell_2} \mathfrak{D}_{\ell_2}(\varphi_1, \chi_1) & \\
& & & & (\varphi_2, \chi_2) & \longrightarrow \quad \cdots
\end{array}
$$

To define the **Arithmetic Wavefront Set** of (φ, χ), we associate each enhanced local L-parameter from the consecutive descents of (φ, χ) a pair

$$(\ell_j, q_{(\ell_j)})$$

with $\ell_j \geq 1$ and $q_{(\ell_j)}$ a one-dimensional $\epsilon_{q_{(\ell_j)}}$-Hermitian form, whose definition is technically involved and is determined uniquely by the rationality of the relevant data.

Arithmetic Wavefront Set of (φ, χ)

Pre-Tableaux: Given a ordered partition $\underline{\ell} = (\ell_1, \ell_2, \ldots, \ell_r)$, we define a pre-tableau $\mathfrak{s}_{\underline{\ell}}$ to be a sequence of pairs:

$$\mathfrak{s}_{\underline{\ell}} := ((\ell_1, q_{(\ell_1)}), (\ell_2, q_{(\ell_2)}), \ldots, (\ell_r, q_{(\ell_r)})).$$

If $(\ell, q_{(\ell)})$ is such a pair, we define

$$(\ell, q_{(\ell)}) \star \mathfrak{s}_{\underline{\ell}} = ((\ell, q_{(\ell)}), (\ell_1, q_{(\ell_1)}), (\ell_2, q_{(\ell_2)}), \ldots, (\ell_r, q_{(\ell_r)}))$$

which is a pre-tableau associated with the partition $(\ell, \ell_1, \ldots, \ell_r)$.

If $\mathfrak{S} = \{\mathfrak{s}_{\underline{\ell}}\}$ is a set of pre-tableaux, then define

$$(\ell, q_{(\ell)}) \star \mathfrak{S}; = \{(\ell, q_{(\ell)}) \star \mathfrak{s}_{\underline{\ell}}\}.$$

If $\ell_1 = 0$ and $q_{(\ell_1)}$ is the zero form, we define

$$(\ell_1, q_{(\ell_1)}) \star \{(\ell, q_{(\ell)})\} := \{(\ell, q_{(\ell)})\}$$

Arithmetic Wavefront Set of (φ, χ)

Pre-Tableaux of (φ, χ)**:** For any (φ, χ), with φ generic and $\chi \in \widehat{\mathcal{S}_\varphi}$, and for $a \in \mathcal{Z}$, the set of pre-tableaux associated with (φ, χ), which is denoted by $\mathcal{T}_a(\varphi, \chi, q)$, is defined inductively via the consecutive descents of (φ, χ) by

$$\mathcal{T}_a(\varphi, \chi, q) := \cup_{0<\ell\leq n} \cup_{z\in\mathcal{Z}} \cup_{(\widehat{\phi}, \widehat{\chi}') \in \mathfrak{D}_\ell^z(\varphi,\chi)} (\mathfrak{l}, q_{(\mathfrak{l})}) \star \mathcal{T}_a(\widehat{\phi}, \widehat{\chi}', q')$$

with compatibility of the rationality conditions involved in the Local Langlands conjecture, the local GGP conjecture and the local descents.

Here $\mathfrak{l} = 2\ell + 1$ if the local descent on the representation side is of Bessel type, and $\mathfrak{l} = 2\ell$ if the local descent on the representation side is of Fourier-Jacobi type.

L-**Descent Pre-Tableaux of** (φ, χ)**:** $\mathcal{L}_a(\varphi, \chi, q)$ is the subset of $\mathcal{T}_a(\varphi, \chi, q)$ consisting of pre-tableaux whose associated partitions are in decreasing order.

Arithmetic Wavefront Set of (φ, χ)

Proposition [JLZ2022]: For any (φ, χ), $\mathcal{L}_a(\varphi, \chi, q)$ is not empty.

By using composition of admissible sesquilinear Young tableaux for $\mathfrak{n}$-dimentional sesquilinear vector spaces (V, q), we obtain a construction

$$\mathcal{L}_a(\varphi, \chi, q) \longleftrightarrow \mathcal{Y}_a(\varphi, \chi, q), \quad \mathfrak{s}_{\underline{\ell}} \mapsto (\underline{p}, \{(V_{(p_j)}, q_{(p_j)})\}).$$

For $\mathfrak{s}_{\underline{\ell}} \in \mathcal{L}_a(\varphi, \chi, q)$, we write the partition of $\mathfrak{n}$ as

$$\underline{\ell} = [\ell_1, \ell_2, \ldots, \ell_r] = [p_1^{m_1}, p_2^{m_2}, \ldots, p_t^{m_t}] = \underline{p},$$

with $p_1 > p_2 > \cdots > p_t$, and

$$(V_{(p_j)}, q_{(p_j)}) = (F^{m_j}, \oplus_{i=1}^{m_j} q_i)$$

satisfying the admissibility conditions.

Arithmetic Wavefront Set of (φ, χ)

(V, q): an $\mathfrak{n}$-dimensional ϵ-Hermitian space, and $G = \mathrm{Isom}(V, q)^\circ$

$\mathcal{Y}(V, q)$: the set of equivalence classes of admissible sesquilinear Young tableaux for (V, q),

Proposition [Gomez-Zhu 2014]: One has a canonical bijection

$$\mathcal{N}_F(\mathfrak{g})_\circ \quad \longleftrightarrow \quad \mathcal{Y}(V, q)$$

with certain technical exceptions when $G = \mathrm{SO}_{2m}$.

Define $\mathrm{WF}(\varphi, \chi) \subset \mathcal{N}_F(\mathfrak{g})_\circ$ that maps to $\mathcal{Y}_a(\varphi, \chi, q) \subset \mathcal{Y}(V, q)$.

Theorem [JLZ2022] If $\pi = \pi(\varphi, \chi) \in \Pi_F(G)$, the set $\mathrm{WF}(\varphi, \chi)$ is completely determined by π.

Arithmetic Wavefront Set

For any $\pi = \pi(\varphi, \chi) \in \Pi_F(G)$ with φ generic, we define the **Arithmetic Wavefront Set** of π by

$$\mathrm{WF}_{\mathrm{ari}}(\pi) := \mathrm{WF}(\varphi, \chi).$$

Wavefront Set (WFS) Conjecture:

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$$

for any $\pi \in \Pi_F(G)$ with generic local L-parameters.

The **WFS Conjecture** $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ can be verified for some special cases: 1) lower rank groups, like SO_7, or 2) special types of representations of classical groups.

It can also be verified for some families of tempered unipotent representations π of SO_{2n+1}, based on the work of Waldspurger (2018, 2019, 2020).

Arithmetic Wavefront Set: $F = \mathbb{R}$

Theorem [Rationality] (JLZ2022): For $F = \mathbb{R}$ and $\varphi \in \Phi_{\mathrm{gen}}(G_n^*)$, if $\pi \in \Pi_\varphi[G_n^*]$, the following hold.

1. If the component group $\mathcal{S}_\varphi$ is trivial, then $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max}$ consists of all the F-rational regular nilpotent orbits in $\mathcal{N}_F(\mathfrak{g})_\circ$.
2. If the component group $\mathcal{S}_\varphi$ is nontrivial, then

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \{\mathcal{O}(\pi)\},$$

   with $\mathcal{O}(\pi) := \mathcal{O}_a(\varphi, \chi)$ if $\pi = \pi_a(\varphi, \chi)$ for some $\chi \in \widehat{\mathcal{S}_\varphi}$, where $\mathcal{O}_a(\varphi, \chi)$ is the unique F-rational nilpotent orbit that is defined via the consecutive descents of (φ, χ).
3. Every $\mathcal{O} \in \mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} \cap \mathcal{N}_F(\mathfrak{g})_\circ$ is F-distinguished in the sense that it does not meet any F-rational proper Levi subalgebras of $\mathfrak{g}_n$.

Arithmetic Wavefront Set: $F = \mathbb{R}$

Theorem (JLZ2022): If $F = \mathbb{R}$, for any $\pi = \pi(\varphi, \chi) \in \Pi_F(G_n)$ with φ generic, the set $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max}$ determines a unique F-stable nilpotent orbit $\mathcal{O}^{\mathrm{st}}(\pi)$ in $\mathcal{N}_F(\mathfrak{g})^{\mathrm{st}}_{\circ}$.

When F is p-adic, this theorem holds for many representations, including all unipotent representations. However, C.-C. Tsai (2022) found an counter-expample of supercuspidal representations of U_7 with $p = 3$ for $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$.

If F is p-adic, we are in progress to understand $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max}$.

A joint work with C. Chen, D. Liu and L. Zhang is to understand applications to the representation side of $\mathrm{WF}_{\mathrm{ari}}(\pi)$, which can be viewed as a progress towards the **Wavefront Set Conjecture**.

Over finite fields, the **Wavefront Set Conjecture** can be verified for all classical groups (Z. Wang-L. Zhang).

Local Descent and Submodule Problem

$G = \mathrm{SO}_{2n+1}$ (same formulation for other classical groups).

Given $\pi \in \Pi_F(G)$, for $1 \leq \ell \leq n$, the twisted Jacquet module $\mathcal{J}_{X_\ell}(\pi)$ is a smooth representation of $M_{\ell,X_\ell}(F)$.

Theorem (Jiang-Zhang (2018)): When F is p-adic and $\pi \in \Pi_F(G)$, if $\mathcal{J}_{X_\ell}(\pi) \neq 0$, then there exists $\sigma \in \Pi_F(M_{\ell,X_\ell})$, s.t.

$$\pi \hookrightarrow \mathrm{Ind}_{R_{\ell,X_\ell}}^{G}(\sigma \otimes \psi_{X_\ell})$$

as a *submodule*, where $R_{\ell,X_\ell} := M_{\ell,X_\ell} \ltimes U_{X_\ell}$.

The idea is to prove: each Bernstein component of $\mathcal{J}_{X_\ell}(\pi)$ is of finite type.

Question: How to prove this type of **Submodule Theorem** when $F = \mathbb{R}$ and $\pi \in \Pi_F(G)$ is tempered and Casselman-Wallach?

Local Descent and Spectrum Problem

From the above discussion of local descents, it suggests that the local descent (or the twisted Jacquet module)

$$\mathcal{D}_{X_{\ell_0}}(\pi) = \mathcal{J}_{X_{\ell_0}}(\pi)$$

has a nicer spectral structure at the first occurrence index $\ell_0(\pi)$.

Jiang-Zhang (2018): If F is p-adic and π has a generic L-parameter, $\mathcal{D}_{X_{\ell_0}}(\pi)$ is a direct sum of discrete series.

Jiang-Soudry (2003); Jiang-Nien-Qin (2010): If F is p-adic, the local descent $\mathcal{D}_{X_{n-1}}(\mathcal{L}(\frac{1}{2}, \tau))$ is an irreducible generic supercuspdal representation.

Question: What can one say about the spectral decomposition of the local descent $\mathcal{D}_{X_{\ell_0}}(\pi)$ at the first occurrence index $\ell_0 = \ell_0(\pi)$ when F is a local field and π is of Arthur type?

Problems on Arithmetic Wavefront Set

Problem (1): For classical groups G, it is an interesting problem to extend the theory of arithmetic wavefront sets for generic Arthur packets to

1. Representations of Arthur type?
2. Representations in the admissible dual?

Problem (2): How to extend the theory of arithmetic wavefront sets to exceptional groups?

Problem (3): What is the theory of local descents for exceptional groups?

Problem (4): What should the analogy of the local Gan-Gross-Prasad conjecture for exceptional groups?

(Thanks!)