Let G be a td group and (π, V) be a smooth G-representation. Fix a (left) Haar measure μ of G and let $C_c^\infty(G) \simeq \mathcal{H}(G)$ denote the Hecke algebra of G.

1. Let $K \subset G$ be a compact, open subgroup.

(a) Show that $e_K * e_K = e_K$.

(b) Show that $\pi(e_K)V = V^K$.

(c) Show that if $\xi \in \mathcal{D}_c(G)$ satisfies that $\xi * e_K \equiv 0$ for each K, then $\xi \equiv 0$.

2. Show that $\mathcal{H}(G)$ has a multiplicative identity if and only if G is discrete.

3. Suppose that

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is a short exact sequence of smooth G-representations. Show that V_2 is admissible if and only if V_1 and V_3 are admissible.

We now recall the context of Friday's lecture. Let $G = F^\times$ for F a non-archimedean local field, and fix a uniformizer $\varpi \in F^\times$. For a fixed character $\omega : \mathcal{O}_F^\times \to \mathbb{C}^\times$, let

$$\Omega(F^\times, \omega) = \{\chi : F^\times \to \mathbb{C}^\times : \chi|_{\mathcal{O}_F^\times} \equiv \omega\}.$$

Recall the map

$$F : \mathcal{D}_c(F^\times) \longrightarrow \mathbb{C}[\Omega(F^\times)],$$

$$\xi \longmapsto (\hat{\xi}_\omega(t))_{\omega \in \Omega(\mathcal{O}_F^\times)},$$

where $\hat{\xi}_\omega(t) = \int_{F^\times} \omega(\arg(x)) t^{\mathrm{val}(x)} \xi(x)$.

4. Let $e_\omega \in \mathcal{H}(F^\times)$ be the function

$$e_\omega(t) = \begin{cases} \omega(t) & : t \in \mathcal{O}_F^\times, \\ 0 & : \text{otherwise.} \end{cases}$$

Prove that

$$F(\mathcal{D}_c(F^\times) * e_\omega) = \mathbb{C}[\Omega(F^\times, \omega)].$$

5. We now complete the proof that the map F extends to an isomorphism

$$F_{ec} : \mathcal{D}_{ec}(F^\times) \xrightarrow{\sim} \mathbb{C}[\Omega(F^\times)],$$

where $\mathcal{D}_{ec}(F^\times) = \{\xi \in C_c^\infty(F^\times)^* : \xi * f \text{ has compact support, } f \in \mathcal{H}(F^\times)\}$ and

$$F_{ec}(\xi) = (F(\xi * e_\omega))_\omega.$$

The preceding exercise shows that it suffices to prove that $\xi \mapsto (\xi * e_\omega)_\omega$ induces an isomorphism

$$\mathcal{D}_{ec}(F^\times) \xrightarrow{\sim} \prod_{\omega \in \Omega(\mathcal{O}_F^\times)} \mathcal{D}_c(F^\times) * e_\omega.$$

(a) Show that for any compact open subgroup $K \subset \mathcal{O}_F^\times$, we have

$$e_K = \sum_{\omega \in \Omega(\mathcal{O}_F^\times/K)} e_\omega.$$

(b) Use this to show that

$$\prod_{\omega \in \Omega(\mathcal{O}_F^\times)} \mathcal{D}_c(F^\times) * e_\omega \simeq \varprojlim_K \mathcal{D}_c(F^\times) * e_K.$$

(c) Prove that F_{ec} induces an isomorphism by showing that $\mathcal{D}_{ec}(F^\times) \simeq \varprojlim_K \mathcal{D}_c(F^\times) * e_K$.

End of assignment